ON SOME OPERATOR IDENTITIES AND REPRESENTATIONS OF ALGEBRAS

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ABSTRACT. Certain infinite families of operator identities related to powers of positive root generators of (super) Lie algebras of first-order differential operators and q-deformed algebras of first-order finite-difference operators are presented.

1. The following *operator identity* holds

$$(J_n^+)^{n+1} \equiv (x^2\partial_x - nx)^{n+1} = x^{2n+2}\partial_x^{n+1}, \partial_x \equiv \frac{d}{dx}, n = 0, 1, 2, \dots \tag{1}$$

The proof is straightforward:

(i) the operator $(J_n^+)^{n+1}$ annihilates the space of all polynomials of degree not higher than n, $\mathcal{P}_n(x) = Span\{x^i : 0 \le i \le n\}$;
(ii) in general, an $(n+1)-$th order linear differential operator annihilating $\mathcal{P}_n(x)$ must have the form $B(x)\partial_x^{n+1}$, where $B(x)$ is an arbitrary function and
(iii) since $(J_n^+)^{n+1}$ is a graded operator, $\deg(J_n^+) = +1$, /footnoteso J_n^+ maps x^k to a multiple of x^{k+1} $\deg(J_n^{+\,n+1}) = n+1$, hence $B(x) = bx^{2n+2}$ while clearly the constant $b = 1$.

It is worth noting that taking the degree in (1) different from $(n+1)$, the l.h.s. in (1) will contain all derivative terms from zero up to $(n+1)$-th order.

The identity has a Lie-algebraic interpretation. The operator (J_n^+) is the positive-root generator of the algebra sl_2 of first-order differential operators (the other sl_2-generators are $J_n^0 = x\partial_x - n/2$, $J_n^- = \partial_x$). Correspondingly, the space $\mathcal{P}_n(x)$ is nothing but the $(n+1)$-dimensional irreducible representation of sl_2. The identity (1) is a consequence of the fact that $(J_n^+)^{n+1} = 0$ in this representation.

There exist other algebras of differential or finite-difference operators (in more than one variable), which admit a finite-dimensional representation. This leads to more general and remarkable operator identities. In the present Note, we show that (1) is one representative of infinite family of identities for differential and finite-difference operators.

2. The Lie-algebraic interpretation presented above allows us to generalize (1) for the case of differential operators of several variables, taking appropriate degrees of the highest-positive-root generators of (super) Lie algebras of first-order differential operators, possessing a finite-dimensional invariant sub-space (see e.g.[1]). First we consider the case of sl_3. There exists a representation of $sl_3(\mathbf{C})$ as differential operators on $\mathbf{C^2}$. One of the generators is

$$J_2^1(n) = x^2\partial_x \;+\; xy\partial_y - nx$$

The space $\mathcal{P}_n(x,y) = Span\{x^iy^j : 0 \le i+j \le n\}$ is a finite-dimensional representation for sl_3, and due to the fact $(J_2^1(n))^{n+1} = 0$ on the space $\mathcal{P}_n(x,y)$, hence we

have

$$(J_2^1(n))^{n+1} = (x^2\partial_x \ + \ xy\partial_y - nx)^{n+1} =$$
$$\sum_{k=0}^{k=n+1} \binom{n+1}{k} x^{2n+2-k} y^k \partial_x^{n+1-k} \partial_y^k \ , \tag{2}$$

This identity is valid for $y \in \mathbf{C}$ (as described above), but also if y is a Grassmann variable, i.e. $y^2 = 0$ [1]. In the last case, $J_2^1(n)$ is a generator of $osp(2,2)$, see [1].

More general (using sl_k instead of sl_3), the following operator identity holds

$$(J_{k-1}^{k-2}(n))^{n+1} \equiv (x_1 \sum_{m=1}^{k} (x_m \partial_{x_m} \ - n))^{n+1} =$$
$$x_1^{n+1} \sum_{j_1+j_2+\ldots+j_k=n+1} C^{n+1}_{j_1,j_2,\ldots,j_k} x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k} \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \ldots \partial_{x_k}^{j_k} \ , \tag{3}$$

where $C^{n+1}_{j_1,j_2,\ldots,j_k}$ are the generalized binomial (multinomial) coefficients. If $x \in \mathbf{C^k}$, then $J_{k-1}^{k-2}(n)$ is a generator of the algebra $sl_k(\mathbf{C})$ [1], while some of the variables x's are Grassmann ones, the operator $J_{k-1}^{k-2}(n)$ is a generator of a certain super Lie algebra of first-order differential operators. The operator in l.h.s. of (3) annihilates the linear space of polynomials $\mathcal{P}_n(x_1, x_2, \ldots x_k) = Span\{x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k} : 0 \le j_1 + j_2 + \ldots + j_k \le n\}$.

3. The above-described family of operator identities can be generalized for the case of finite-difference operators with the Jackson symbol, D_x (see e.g. [2])

$$D_x f(x) = \frac{f(x) - f(q^2 x)}{(1-q^2)x} + f(q^2 x) D_x$$

instead of the ordinary derivative. Here, q is an arbitrary complex number. The following operator identity holds

$$(\tilde{J}_n^+)^{n+1} \equiv (x^2 D_x - \{n\}x)^{n+1} = q^{2n(n+1)} x^{2n+2} D_x^{n+1}, n = 0, 1, 2, \ldots \tag{4}$$

(cf.(1)), where $\{n\} = \frac{1-q^{2n}}{1-q^2}$ is so-called q-number. The operator in the r.h.s. annihilates the space $\mathcal{P}_n(x)$. The proof is similar to the proof of the identity (1).

From algebraic point of view the operator $\tilde{J}_n^+$ is the generator of a q-deformed algebra $sl_2(\mathbf{C})_{\mathbf{q}}$ of first-order finite-difference operators on the line: $\tilde{J}_n^0 = xD - \hat{n}$, $\tilde{J}_n^- = D$, where $\hat{n} \equiv \frac{\{n\}\{n+1\}}{\{2n+2\}}$ (see [3] and also [1]), obeying the commutation relations

$$\begin{aligned} q^2 \tilde{j}^0 \tilde{j}^- \ - \ \tilde{j}^- \tilde{j}^0 \ &= \ -\tilde{j}^- \\ q^4 \tilde{j}^+ \tilde{j}^- \ - \ \tilde{j}^- \tilde{j}^+ \ &= \ -(q^2+1)\tilde{j}^0 \\ \tilde{j}^0 \tilde{j}^+ \ - \ q^2 \tilde{j}^+ \tilde{j}^0 \ &= \ \tilde{j}^+ \end{aligned} \tag{5}$$

($\tilde{j}$'s are related with $\tilde{J}$'s through some multiplicative factors). The algebra (5) has the linear space $\mathcal{P}_n(x)$ as a finite-dimensional representation.

An attempt to generalize (2) replacing continuous derivatives by Jackson symbols immediately leads to requirement tp introduce the quantum plane and q-differential calculus [4]

$$xy = qyx \ ,$$

[1] In this case just two terms in l.h.s. of (2) survive.

$$D_x x = 1 + q^2 x D_x + (q^2-1) y D_y \quad , \quad D_x y = q y D_x \ ,$$
$$D_y x = q x D_y \quad , \quad D_y y = 1 + q^2 y D_y \ ,$$
$$D_x D_y = q^{-1} D_y D_x \ . \tag{6}$$

The formulae analogous to (2) have the form

$$(\tilde{J}^1_2(n))^{n+1} \equiv (x^2 D_x \ + \ xy D_y - \{n\} x)^{n+1} =$$
$$\sum_{k=0}^{k=n+1} q^{2n^2 - n(k-2) + k(k-1)} \binom{n+1}{k}_q x^{2n+2-k} y^k D_x^{n+1-k} D_y^k \ , \tag{7}$$

where

$$\binom{n}{k}_q \equiv \frac{\{n\}!}{\{k\}!\{n-k\}!} \ , \ \{n\}! = \{1\}\{2\}\ldots\{n\}$$

are q-binomial coefficient and q-factorial, respectively. Like all previous cases, if $y \in \mathbf{C}$, the operator $\tilde{J}^1_2(n)$ is one of generators of q-deformed algebra $sl_3(\mathbf{C})_{\mathbf{q}}$ of finite-difference operators, acting on the quantum plane and having the linear space $\mathcal{P}_n(x,y) = Span\{x^i y^j : 0 \le i+j \le n\}$ as a finite-dimensional representation; the l.h.s. of (7) annihilates $\mathcal{P}_n(x,y)$. If y is Grassmann variable, $\tilde{J}^1_2(n)$ is a generator of the q-deformed superalgebra $osp(2,2)_q$ possessing finite-dimensional representation.

Introducing a quantum hyperplane [4], one can generalize the whole family of the operator identities (3) replacing continuous derivatives by finite-difference operators.

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